CS5283 Lecture 14: Queueing Networks, and Stochastic Process Review

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Ergodicity

- ⋆ **For any stochastic process** $x(t)$ **we can have two averages: (a) Time averages & (b) Ensemble averages**
- ⋆ **Ensemble Averages: Fix** t **at some** C**. Then** $E[x(C)] = \mu_x(C) = \int_{-\infty}^{+\infty} x f(x; C) dx$
- ⋆ $\mu_x(C)$ **is the average of** $x(t)$ **for a particular** t **over all ensembles. Hence the name.**
- ⋆ **Time Averages: Fix a particular ensemble, and find the average values of** $x(t)$**; In other words, it is the average value of** $x(t)$ **over time**
- ⋆ $M_T = \frac{1}{2T} \int_{-T}^{T} x(t) dt$**. Notice that** M_T **is a**

random variable, because calculation of M_T for a single time function, is calculation for a single outcome.

- ⋆ **For a stationary process, we have** $E[M_T] = E[\frac{1}{2T}\int_{-T}^{T} x(t)dt] = \frac{1}{2T}\int_{-T}^{T} E[x(t)]dt = \mu$**.**
- ⋆ **A stationary process is said to be ergodic if the time average equals the ensemble average.**
- ⋆ **In other words, a stationary process is ergodic if $Var(M_T) = 0$, as $T \to \infty$.**

Networks of Queues

- ⋆ **Often queues are inter-connected forming a network!**
- ⋆ **The queues can be networked in any fashion with traffic splits, & merges**
- ⋆ **No exact method for analyzing such generalized queueing networks**
- ⋆ **Exact and simple solution exists only if the traffic flow in Poisson and service times are exponential**

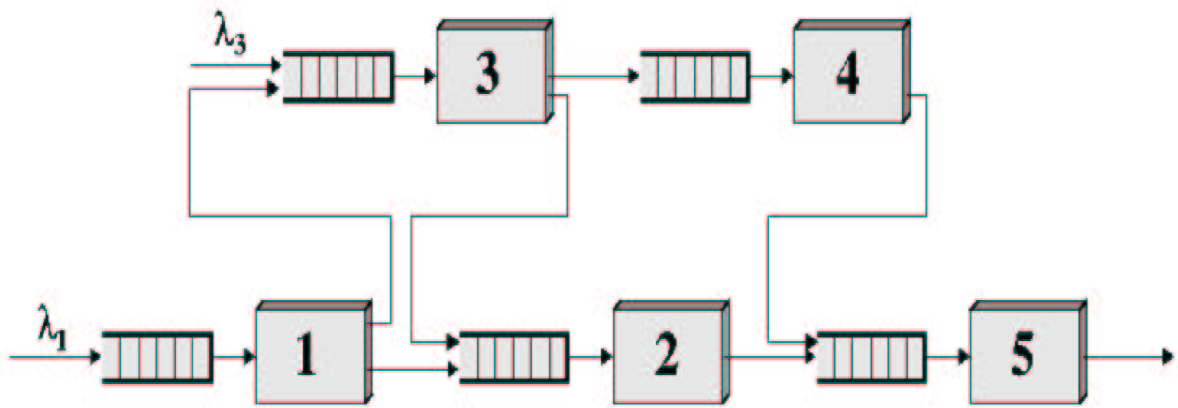


Figure 8.7 Example of a Network of Queues

Traffic Splits, Merges and Tandem

- ⋆ Poisson traffic split by uniform probability remains Poisson
- ⋆ Merging of two Poisson traffic streams results in a Poisson stream
- ⋆ In Tandem infinite queues with exponential service times, if the input traffic is Poisson, then the input to each subsequent queue is also Poisson and identical to the original input!
- ⋆ Hence each queue can be analyzed independently

- $\star$ **Can be extended to multi-server queues also**

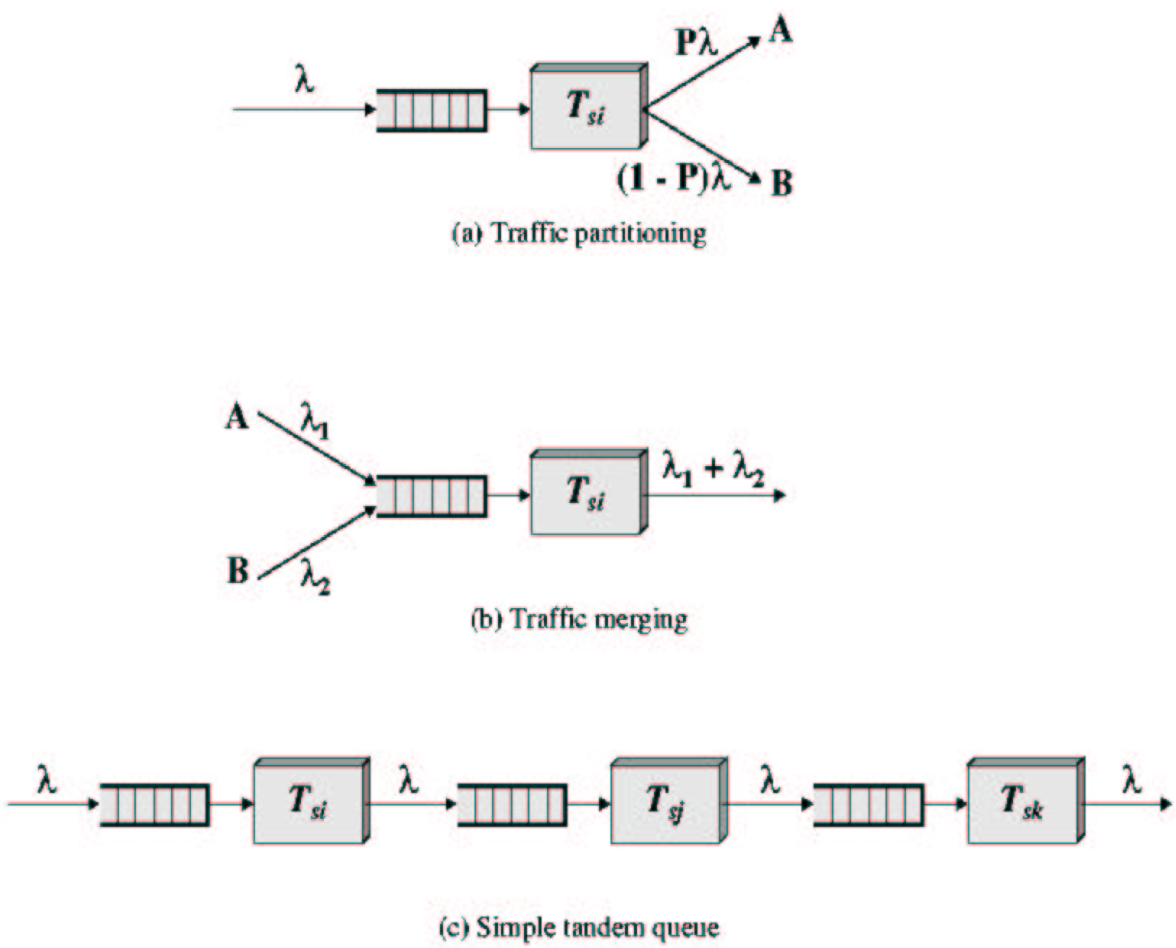


Figure 8.8 Elements of Queuing Networks

Jackson's Theorem

Consider a queueing network with m nodes

- ⋆ **Each node provides independent exponential service**
- ⋆ **Items arriving from outside into any of the nodes, do as per a Poisson process**
- ⋆ **Once served at a node, an item immediately goes to another node with a fixed probability or outside the system**

In such a network, each node is an independent queueing system & can be analyzed independently using the M/M/1 or M/M/N models.

Results can be merged together. Mean system delays is the sum of individual delays, but not so for higher order moments.

Can this be applied for studying packet switched networks?

Packet-Switching network

Consider a queueing network with N sources and destinations. Let

- $\star$ L **be the number of links in the network.**
- $\star$ R_i **be the transmission rate of link** i**.**
- $\star$ M **be the average packet size**
- $\star$ γ_{jk} **be the load between source** j **and destination** k**.**

Determine the average path delay T experienced by a packet through the network.

Packet-Switching network

- **Total external load offered to the network,** $\gamma = \sum_{j=1}^{N} \sum_{k=1}^{N} \gamma_{jk}$
- **Let** π_{jk} **denote the path from node** j **to node** k**. The total load on link** i **is then given by**

$$\lambda_i = \sum_{j,k:\ i \in \pi_{jk}} \sum \gamma_{j,k}$$

- **This implies that the analysis is valid only for fixed routing; Not for adaptive/dynamic routing.**
- **Intuitively,** $E[\text{number of links in a path}] = \frac{\sum \lambda_i}{\gamma}$

- **By Little's law, Average # of packets waiting at link** i**,** $r_i = \lambda_i T_{ri}$**.**
- **Applying Little's Law to the entire network,** $R = \gamma T$**.**
- **Notice that sum of averages is the average of the sum. Hence, equating the above two expressions, we get,** $T = \frac{1}{\gamma} \sum_{i=1}^{L} \lambda_i T_{ri}$
- **By Jackson's theorem, we can treat each node independently as a M/M/1 queue. Hence,** $T_{ri} = \frac{T_{si}}{1-\rho_i} = \frac{T_{si}}{1-\lambda_i T_{Si}}$
- T_{si} **is nothing but** M/R_i**. Thus,** $T = \frac{1}{\gamma} \sum_{i=1}^{L} \frac{M\lambda_i}{R_i - M\lambda_i}$

Review Problems

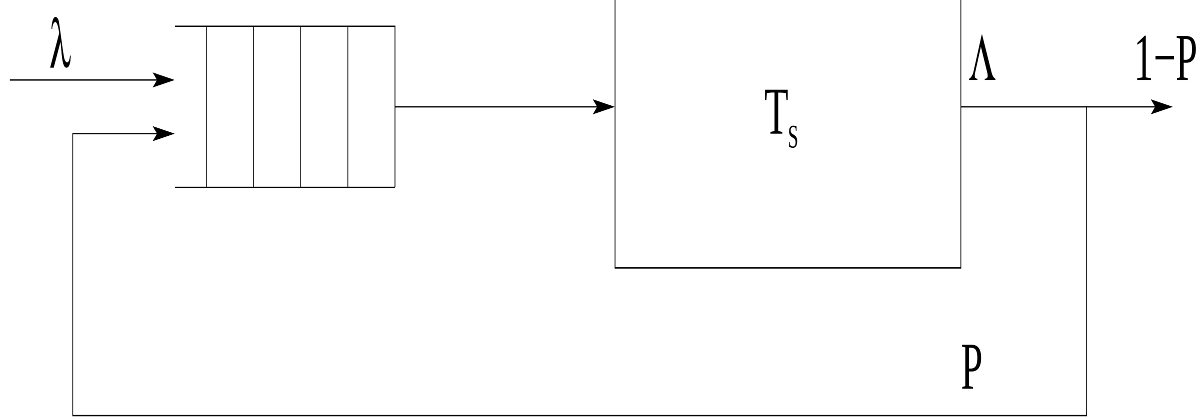
1. **Let** X **and** Y **be two random variables such that,** $E[X^2] = E[Y^2] = E[XY]$**. Show that** $X = Y$**.**

2. **Find** $E[X^2 Y]$**, where** X **is a zero-mean, unit-variance Gaussian random variable, and** Y **is an uniform random variable in the interval** $[-1, 3]$**. Assume** X **and** Y **to be independent.**

3. **Let** $X(t) = cos(wt + \theta)$**, where** θ **is uniformly distributed in the interval** $(-\pi, \pi)$**. Is** $X(t)$ **Stationary?**

4. A fraction P of the traffic from a single exponential server is fed back into the input as shown in the figure.

- $\star$ **Determine the system throughput Λ, server utilization, and the mean residence time for one pass through the server.**
- $\star$ **Determine the mean number of passes that an item makes through the system, and the mean total time spent in the system.**


λ
T_S
Λ
1−P
P