Regular and Nonregular Languages

a^*b^* is regular.

$\{a^nb^n : n \geq 0\}$ is not.

$\{w \in \{a, b\}^* :$ every a is immediately followed by b$\}$ is regular.

$\{w \in \{a, b\}^* :$ every a has a matching b somewhere$\}$ is not

Questions:

- Showing that a language is regular.
- Showing that a language is not regular.

Showing that a Language is Regular

Theorem: Every finite language is regular.

Proof: If L is the empty set, then it is defined by the regular expression $\varnothing$ and so is regular. If it is any finite language composed of the strings s_1, s_2, … s_n for some positive integer n, then it is defined by the regular expression:

$$s_1 \cup s_2 \cup \ldots \cup s_n$$

So it too is regular.

Showing that a Language is Regular

Example:

Let $L = L_1 \cap L_2$, where:

$L_1 = \{a^n b^n, n \geq 0\}$, and

$L_2 = \{b^n a^n, n \geq 0\}$

L_1 and L_2 are infinite however

$L = \{ \varepsilon \}$ is regular

Showing that a Language is Regular

1. Show that *L* is finite.

2. Exhibit an FSM for *L*.

3. Exhibit a regular expression for *L*.

4. Show that the number of equivalence classes of $\approx_L$ is finite.

5. Exhibit a regular grammar for *L*.

6. Exploit the closure theorems

Closure Properties of Regular Languages

- Union
- Concatenation
- Kleene star
- Complement
- Intersection
- Difference
- Reverse
- Letter substitution

Letter Substitution

- Let Σ_1 and Σ_2 be alphabets.

- Let *sub* be any function from Σ_1 to Σ_2^*.

Example:

Let: $\Sigma_1 = \{\texttt{a}, \texttt{b}\}$,
$\Sigma_2 = \{\texttt{0}, \texttt{1}\}$,

sub(a) = 0, and
sub(b) = 11.

Letter Substitution

- *letsub* is a letter substitution function iff:

$letsub(L_1) = \{w \in \Sigma_2^* : \exists y \in L_1$ and
$w = y$ except that:
every character c of y
is replaced by $sub(c)\}$.

Example:

$sub(\mathtt{a}) = \mathtt{0}$, and
$sub(\mathtt{b}) = \mathtt{11}$.

Then $letsub(\{\mathtt{a}^n\mathtt{b}^n, n \geq 0\}) = 0^n1^{2n}$

Showing that a Language is Not Regular

Every regular language can be accepted by some FSM.

It can only use a finite amount of memory to record essential properties.

Example:

$\{a^n b^n, n \geq 0\}$ is not regular

Showing that a Language is Not Regular

The only way to generate/accept an infinite language with a finite description is to use:

Kleene star (in regular expressions), or

cycles (in automata).

This forces some kind of simple repetitive cycle within the strings.

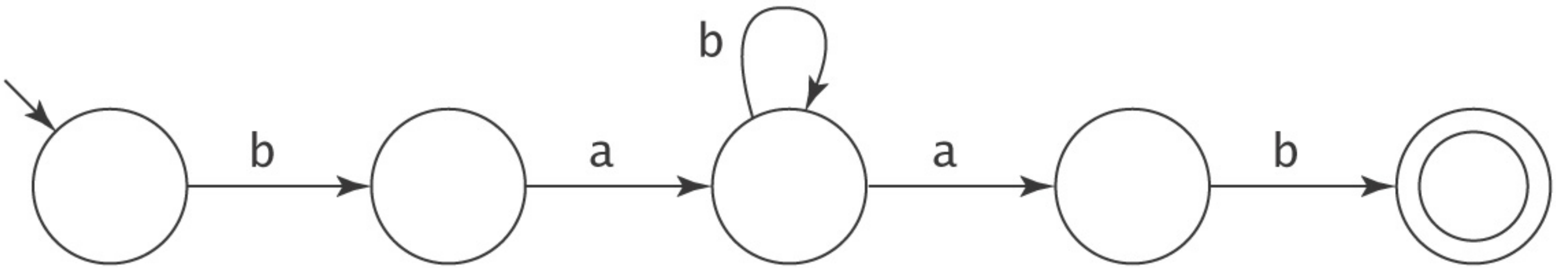
Example:

`ab*a` generates `aba`, `abba`, `abbba`, `abbbba`, etc.

Example:

$\{a^n : n \geq 1$ is a prime number$\}$ is not regular.

Exploiting the Repetitive Property



If an FSM with *n* states accepts any string of length ≥ *n*, how many strings does it accept?

L = `bab*ab`

`b a`	`b`	`b b b a b`
x	*y*	*z*

*xy*z* must be in *L*.

So *L* includes: `baab`, `babab`, `babbab`, `babbbbbbbbbbab`

Theorem – Long Strings

Theorem: Let $M = (K, \Sigma, \delta, s, A)$ be any DFSM. If M accepts any string of length $|K|$ or greater, then that string will force M to visit some state more than once (thus traversing at least one loop).

Proof: M must start in one of its states. Each time it reads an input character, it visits some state. So, in processing a string of length n, M creates a total of $n + 1$ state visits. If $n+1 > |K|$, then, by the pigeonhole principle, some state must get more than one visit. So, if $n \geq |K|$, then M must visit at least one state more than once.

The Pumping Theorem for Regular Languages

If L is regular, then every long string in L is pumpable.

So, $\exists k \geq 1$

($\forall$ strings $w \in L$, where $|w| \geq k$

($\exists$ x, y, z ($w = xyz$,

$|xy| \leq k$,

$y \neq \varepsilon$, and

$\forall q \geq 0$ (xy^qz is in L)))).

Example: $\{a^n b^n : n \geq 0\}$ is not Regular

If L were regular, then there would exist some k such that any string w where $|w| \geq k$ must satisfy the conditions of the theorem. Let $w = a^{\lceil k/2 \rceil} b^{\lceil k/2 \rceil}$. Since $|w| \geq k$, w must satisfy the conditions of the pumping theorem. So, for some x, y, and z, $w = xyz$, $|xy| \leq k$, $y \neq \varepsilon$, and $\forall q \geq 0$, xy^qz is in L. We show that no such x, y, and z exist. There are 3 cases for where y could occur: We divide w into two regions:

aaaaa.....aaaaaa	bbbbb.....bbbbbb
1	2

So y can fall in:

- (1): $y = a^p$ for some p. Since $y \neq \varepsilon$, p must be greater than 0. Let $q = 2$. The resulting string is $a^{k+p}b^k$. But this string is not in L, since it has more a's than b's.
- (2): $y = b^p$ for some p. Since $y \neq \varepsilon$, p must be greater than 0. Let $q = 2$. The resulting string is $a^k b^{k+p}$. But this string is not in L, since it has more b's than a's.
- (1, 2): $y = a^p b^r$ for some non-zero p and r. Let $q = 2$. The resulting string will have interleaved a's and b's, and so is not in L.

There exists one long string in L for which no x, y, z exist. So L is not regular

Using the Pumping Theorem

If L is regular, then every "long" string in L is pumpable.

To show that L is not regular, we find one that isn't.

To use the Pumping Theorem to show that a language L is not regular, we must:

1. Choose a string w where $|w| \geq k$. Since we do not know what k is, we must state w in terms of k.
2. Divide the possibilities for y into a set of equivalence classes that can be considered together.
3. For each such class of possible y values where $|xy| \leq k$ and $y \neq \varepsilon$:

   Choose a value for q such that xy^qz is not in L.

Example: $L = \{a^n : n \text{ is prime}\}$

Let $w = a^j$, where j is the smallest prime number $> k+1$.

$y = a^p$ for some p.

$\forall q \geq 0$ ($a^{|x| + |z| + q|y|}$ must be in L). So $|x| + |z| + q \cdot |y|$ must be prime.

But suppose that $q = |x| + |z|$. Then:

$$|x| + |z| + q \cdot |y| = |x| + |z| + (|x| + |z|) \cdot y$$
$$= (|x| + |z|) \cdot (1 + |y|),$$

which is non-prime if both factors are greater than 1:

$(|x| + |z|) > 1$ because $|w| > k+1$ and $|y| \leq k$.
$(1 + |y|) > 1$ because $|y| > 0$.

Using the Closure Properties

The two most useful ones are closure under:

Intersection

Complement

Using the Closure Properties

The two most useful ones are closure under:

Intersection and Complement

Example:

$L = \{w \in \{a, b\}^*: \#_a(w) = \#_b(w)\}$

If L were regular, then:

$L' = L \cap a^*b^*$

would also be regular. But it isn't.

Using the Closure Properties

$L = \{a^i b^j : i, j \geq 0 \text{ and } i \neq j\}$

Try to use the Pumping Theorem by letting $w = a^k b^{k+k!}$.

Then $y = a^p$ for some nonzero p.

Let $q = (k!/p) + 1$ (i.e., pump in $(k!/p)$ times).

Note that $(k!/p)$ must be an integer because $p < k$.

The number of a's in the new string is $k + (k!/p)p = k + k!$.

So the new string is $a^{k+k!} b^{k+k!}$, which has equal numbers of a's and b's and so is not in L.

Using the Closure Properties

$L = \{a^i b^j : i, j \geq 0 \text{ and } i \neq j\}$

An easier way:

If L is regular then so is $\neg L$. Is it?

$\neg L = a^n b^n \cup$ {out of order}

If $\neg L$ is regular, then so is $L' = \neg L \cap a^*b^*$

$= a^n b^n$

Using the Closure Properties

$L = \{a^i b^j c^k : i, j, k \geq 0 \text{ and } (i \neq 1 \text{ or } j = k)\}$

Every string in *L* of length at least 1 is pumpable:

If *i* = 0 then: if *j* ≠ 0, let *y* be `b`; otherwise, let *y* be `c`. Pump in or out. Then *i* will still be 0 and thus not equal to 1, so the resulting string is in *L*.

If *i* = 1 then: let *y* be `a`. Pump in or out. Then *i* will no longer equal 1, so the resulting string is in *L*.

If *i* = 2 then: let *y* be `aa`. Pump in or out. Then *i* cannot equal 1, so the resulting string is in *L*.

If *i* > 2 then: let *y* = `a`. Pump out once or in any number of times. Then *i* cannot equal 1, so the resulting string is in *L*.

Using the Closure Properties

$L = \{a^i b^j c^k : i, j, k \geq 0 \text{ and } (i \neq 1 \text{ or } j = k)\}$

Suppose we guarantee that $i = 1$. If L is regular, then so is:

$L' = L \cap ab^*c^*$.

$L' = \{ab^j c^k : j, k \geq 0 \text{ and } j = k\}$

Use Pumping to show that L' is not regular:

OR

If L is regular, then so is L^R:

$L^R = \{c^k b^j a^i : i, j, k \geq 0 \text{ and } (i \neq 1 \text{ or } j = k)\}$

Use Pumping to show that L' is not regular: