1. [20 points] Prove or disprove each of the following statements.

(a) [5 points] Let $M_1$ be a nondeterministic finite automaton without $\epsilon$-transitions, and all of the states of $M_1$ are reachable from its start state. Construct the equivalent deterministic finite automaton $M_2$ ($L(M_2) = L(M_1)$) via the Subset Construction, and $M_2$ contains only states reachable from its start state.

Then $M_2$ must have at least as many states as $M_1$. 
(b) [5 points] Let $M_1$ be an $n$-state nondeterministic finite automaton without $\epsilon$-transitions, and $M_1$ does not have more than one transition from any state on the same symbol. Then there exists an equivalent deterministic finite automaton $M_2$ ($L(M_2) = L(M_1)$) with exactly $n + 1$ states.
(c) [5 points] Let $M$ be a pushdown automaton such that it never, when reading any input string, can have more than three total stack symbols in its stack (that is, stack-height is at most three).

Then the language $L(M)$ is regular.
(d) [5 points] The following language $L$:

$$L = \{x \in \{a, b, c\}^* \mid \#_a(x) = \#_b(x) + \#_c(x)\}$$

(where $\#_u(w)$ denotes the number of occurrences of the symbol $u$ in the string $w$) is context-free.
2. [15 points] Let the alphabet $\Sigma = \{E, S, W, N\}$ represent the one-unit movements of a particle to the East, South, West, or North direction, respectively.
The language $R$ ("returning to origin") is the set of all strings over $\Sigma$ such that if a particle follows the movement-symbols given by the string, on a large flat field, will return to the original starting position.

(a) [6 points] Prove or disprove that $R$ is a regular language.
(b) [6 points] Prove or disprove that $R$ is a context-free language.
(c) [3 points] Prove or disprove that $R$ is the intersection of two context-free languages.
3. [20 points]

(a) [8 points] Consider the following language $L$:

$$L = \{ \langle T_1, T_2 \rangle \mid T_1 \text{ and } T_2 \text{ are deterministic Turing machines and there exists a string } x \text{ such that both } T_1 \text{ and } T_2 \text{ accept } x \text{ and } T_1 \text{ does so in fewer steps than does } T_2 \}. $$

Show that $L$ is Turing-recognizable by constructing a Turing machine that recognizes $L$. Does your Turing machine halt on all inputs? Justify your answer.
(b) [8 points] Consider the following language \( \text{SUB}_{\text{CFG}, \text{DFA}} \):

\[
\text{SUB}_{\text{CFG}, \text{DFA}} = \{(G, M) \mid G \text{ is a context-free grammar and } M \text{ is a deterministic finite automaton such that } L(G) \subseteq L(M)\}.
\]

Prove that \( \text{SUB}_{\text{CFG}, \text{DFA}} \) is Turing-decidable by constructing a Turing machine that decides \( \text{SUB}_{\text{CFG}, \text{DFA}} \).
(c) [4 points] Prove or disprove that there exists a countably infinite family \( \{X_1, X_2, \ldots \} \) of countably infinite sets \( X_1, X_2, \ldots \) such that for all positive integers \( i \) and \( j \) with \( i \neq j \), the intersection \( X_i \cap X_j \) is a non-empty finite set.