1. Read the notes above carefully.

2. For each of the following languages, construct a finite automaton (deterministic, nondeterministic, or nondeterministic finite automaton with ε-transitions — unless unless specifically stated) that accepts the language.

   You need to give the key idea(s) for your construction, and brief and precise interpretations for the states of the machine.

   (a) [ Construction of “deterministic finite automaton” is required. ]

   \[ \{ x \in \{0, 1\}^* \mid \#_0(x) = \#_1(x) \text{ and every prefix of } x \text{ has at most one more 0 than } 1s \text{ and at most one more } 1 \text{ than } 0s \} \].

   (Note: \#_u(v) denotes the number of occurrences of a substring \( u \) in a string \( v \).)

   (b) [ Construction of “deterministic finite automaton” is required. ]

   \[ \{ a^ib^j \mid i, j \geq 0, \text{ and } i + j \text{ is even} \}. \]

   (c) [ Construction of “deterministic finite automaton” is required. ]

   \[ \{ x \in \{0, 1, 2\}^* \mid \#_1(x) + \#_2(x) \text{ is divisible by } 3 \}. \]

   (d) \( \{ x \in \{0, 1\}^* \mid \text{ there exist two } 0s \text{ in } x \text{ that are separated by a string of length } 5k \text{ for some } k \geq 0 \} \).

   (e) The set of all strings over the alphabet \( \{a, b, c\} \) that yield the same value when evaluated from left to right as right to left by “multiplying” according to the following table in Figure 1.

   For examples: \((a \circ b) \circ b = (c \circ b) = a \text{ and } (a \circ (b \circ b)) = (a \circ a) = a\), whereas \((a \circ b) \circ c) = (c \circ c) = b \text{ and } (a \circ (b \circ c)) = (a \circ c) = c.\)

   \[
   \begin{array}{c|ccc}
   \circ & a & b & c \\
   \hline
   a & a & c & c \\
   b & b & a & c \\
   c & c & a & b \\
   \end{array}
   \]

   Figure 1: A non-associative multiplication table for \( \circ \).

3. Prove that there does not exist any deterministic finite automaton that accepts the following language:

   \[ \{ab^n a^{2n} \mid n \geq 1 \}. \]
4. Consider a nondeterministic finite automaton $M_1 = (Q, \Sigma, \delta, q_0, F_1)$. Define a (new) nondeterministic finite automaton $M_2 = (Q, \Sigma, \delta, q_0, F_2)$ with $F_2 = Q - F_1$.

Prove, or disprove (with explicit counter-example and detailed explanation), the following statement: the language $L(M_2)$ is the complement of the language $L(M_1)$ (that is, $L(M_2) = \Sigma^* - L(M_1)$).

5. Let the alphabet $\Sigma = \{a\}$. Assume that that $M$ is a nondeterministic finite automaton with $m$ states such that $M$ accepts every string $x \in \Sigma^*$ with $|x| \leq m$.

Prove, or disprove (with explicit counter-example and detailed explanation), the following statement: $L(M) = \Sigma^*$.

6. Convert the following nondeterministic finite automaton with $\epsilon$-transitions (see figure below), $M$, to an equivalent nondeterministic finite automaton $M_1$, and then using the Subset Construction to convert $M_1$ to an equivalent deterministic finite automaton $M_2$ with its inaccessible states removed.

Explicitly and briefly write down each step which you perform, such as:

(a) Computing all the $\epsilon$-closures of the states of $M$ (the notation $E(\cdot)$ introduced in page 56 of [sip12]), and
(b) showing complete state-transition diagrams of $M_1$ and $M_2$.

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![Finite Automaton Diagram]

- $q_0$ to $q_1$ on $\epsilon$
- $q_1$ to $q_2$ on $1$
- $q_0$ to $q_1$ on $0$
- $q_1$ to $q_0$ on $\epsilon$
- $q_2$ to $q_0$ on $0$
- $q_0$ to $q_2$ on $1$