Notes:

- Read Course Information: Section 7 (Miscellaneous) and Section 9 (Academic Dishonesty or Misconduct).
- When you are giving a construction, example, etc., provide a justification with your argument. Your solutions to numerical problems must contain the derivation of your answers. In all of your presentations, strive for correctness, completeness, and clarity. When in doubt about the assumptions of problems, the interpretations of wording, etc., consult the instructor.
- You should strive to complete all problems assigned, and a subset of them will be graded.

1. Read the notes above carefully.
2. Read [Sip12] Chapter 0; you may need to review the prerequisite materials in discrete mathematics to have sufficient working knowledge for this course.
3. Do [Sip12] Chapter 0, problem 0.13.
4. An alphabet is a non-empty finite set of symbols, and a string over the alphabet is a finite sequence of symbols of the alphabet. Some example strings over the binary alphabet \{0, 1\} are: 1011 (for the sequence (1, 0, 1, 1), 10 (for the sequence (1, 0)), \(\epsilon\) (denoting the empty sequence).

For strings \(x\) and \(y\) over an alphabet, we denote by \(|x|\) the length of the sequence \(x\), and by \(xy\) the concatenation of the two sequences \(x\) and \(y\) in that order.

For each integer \(n \geq 0\), we define the strings \(x_n\) and \(y_n\) over the alphabet \{0, 1\} as follows: \(x_0 = 0\) and \(y_0 = 1\), and for \(n \geq 1\), \(x_n = x_{n-1}y_{n-1}\) and \(y_n = y_{n-1}x_{n-1}\). Prove the following statements using mathematical induction:

(a) For every \(n \geq 0\), \(|x_n| = |y_n|\).
(b) For every \(n \geq 0\), \(x_n\) and \(y_n\) differ in every position.
(c) For every \(n \geq 0\), \(x_{2n}\) and \(y_{2n}\) are palindromes. (A string \(x\) is a palindrome if the reversal sequence of \(x\) is identical to the sequence \(x\).)
(d) For every \(n \geq 0\), \(x_n\) contains neither the substring 000 nor the substring 111. (A string \(x\) is a substring of a string \(y\) if \(x\) is simply a contiguous subsequence of \(y\).)

5. Let \(\Sigma\) be an alphabet. For a string \(x\) over \(\Sigma\), the reversal of \(x\), denoted by \(x^r\), is the string given the the reverse sequence of \(x\). In lectures, we have defined the string-concatenation operator \(\circ\) and the string-length function \(|\cdot|\) inductively. Give an inductive definition of the string-reversal operator \(^r\).

Prove the following statements using mathematical induction — based on the inductive definitions of \(\circ\), \(^r\), and \(|\cdot|\):

(a) For every string \(x \in \Sigma^*, \ (x^r)^r = x\).
(b) For all strings \(x, y \in \Sigma^*\), \(|xy| = |x| + |y|\).

6. Give some examples of strings in, and not in, the following languages over the alphabet \(\Sigma = \{a, b\}\). Explain your answers.

(a) \(\{w \in \Sigma^* \mid \text{for some } u \in \Sigma^2, w = uu^ru\}\).
(b) \(\{w \in \Sigma^* \mid w w = www\}\).
(c) \(\{w \in \Sigma^* \mid \text{for some strings } u \text{ and } v \text{ over } \Sigma, uvw = wvu\}\).
(d) \(\{w \in \Sigma^* \mid \text{for some string } u \text{ over } \Sigma, uvw = wu\}\).

7. Prove, or disprove (via a counter example), each of the following statements:

(a) For all languages \(L_1, L_2, L_3\) over an alphabet, \((L_1 \cup L_2)L_3 \subseteq (L_1L_3) \cup (L_2L_3)\).
(b) For all languages \(L_1, L_2, L_3\) over an alphabet, \((L_1L_3) \cup (L_2L_3) \subseteq (L_1 \cup L_2)L_3\).
(c) For all languages \(L_1, L_2, L_3\) over an alphabet, \((L_1 - L_2)L_3 \subseteq (L_1L_3) - (L_2L_3)\).
(d) For all languages \(L_1, L_2, L_3\) over an alphabet, \((L_1L_3) - (L_2L_3) \subseteq (L_1 - L_2)L_3\).