Problem 2:

(a) No consecutive 1s, allowing for no consecutive 1s.

(b) 0 or more 1s. 0 must be followed by either a 0 or at least two 1s before another 0.

(c) (01^+001^+0000 (1^0+1^00) 1

(d) A segment of 0's, followed by 1, and with any number of 0s between.
Problem 2

(a) We prove that \((s+r)^* = r^*(s^*r^*)^*\)

(b) To show that the set of all palindromes is a regular language, we apply

By induction on \(n\) (Then this implies the desired language—equally).

Here we elect to follow the approach. Showing:

Let \(x \in (L \cup L(n))^*\). Then \(x = yz\) with \(y \in L(n) \cup L(n)^*\) and \(z \in L(r)^*\).

Case when \(y = e\): Then \(y = y\)...

Case when \(y \neq e\): Then \(y = y_1,\ldots, y_n\) for some \(n \geq 1\) and \(y_i \in \{1, 2, \ldots, n\}\) with \(y_i \neq e\) for some \(i \neq 1\).

So, for each \(i = 1, 2, \ldots, n\)

\(y_i = u_i \cup v_i\) where \(u_i \in (L(n) \cup L(n))^*\) and \(v_i \in L(r)^*\),

or \(y_i = v_i\) where \(v_i \in L(r)^*\).
This says that
\[ x = y_1 y_2 \ldots y_n z \in L(r)^n \cdot L(s) L(r^* U L(r)) \cdot \]

Therefore \( x \in L(r) \cdot L(s) L(r) U L(r) \).

Thus, \( L(r) L(s) U L(r) \cdot L(r^*) \subseteq L(r) \cdot L(s) L(r) U L(r) \).

Similar argument can show the reverse subset containment.

Therefore, \( (rs + r)^* r = r (sr + r^*) \).

(b) \( (r+s)^* \neq r^* + s^* \) in general.

counter-example: \( r= a \) and \( s = b \) \( (\Sigma = \{ a, b \}) \)

\((r+s)^* \) denotes the language \( \{ a, b \}^* \)

and \( r^* + s^* \) denotes the language \( \{ a \}^* \cup \{ b \}^* \).

(c) \( (r^* s^*)^* = (r+s)^* \).

True; prove it.
4

(a) Let \( \alpha \leq 0 \) and \( \beta \geq 0 \). Show that \( \gamma(n) = \alpha n + \beta \) is not regular. Let \( \gamma(n) = \alpha n^2 + \beta n + \gamma \). Consider all possible \( \alpha, \beta, \gamma \). There is one case where \( \gamma(n) \) is not regular.

\[ \gamma(n) = \alpha n^2 + \beta n + \gamma \]

(b) Let \( \alpha > 0 \) and \( \beta > 0 \). Show that \( \gamma(n) = \alpha n^2 + \beta n + \gamma \) is not regular. Let \( \gamma(n) = \alpha n^2 + \beta n + \gamma \). Consider all possible \( \alpha, \beta, \gamma \). There is one case where \( \gamma(n) \) is not regular.

\[ \gamma(n) = \alpha n^2 + \beta n + \gamma \]

(c) Let \( \alpha < 0 \) and \( \beta < 0 \). Show that \( \gamma(n) = \alpha n^2 + \beta n + \gamma \) is not regular. Let \( \gamma(n) = \alpha n^2 + \beta n + \gamma \). Consider all possible \( \alpha, \beta, \gamma \). There is one case where \( \gamma(n) \) is not regular.

\[ \gamma(n) = \alpha n^2 + \beta n + \gamma \]
(b) \( L_2 = \{ a^{i^3} \mid i \geq 0 \} \) is not regular - by Pumping Lemma.

Suppose that \( L_2 \) were regular.

Let \( n \) be the constant in Pumping Lemma.

Consider \( z = a^{n^3} \in L_2 \) with \( |z| = n^3 \geq n \).

We may develop a contradiction similar to that for proving the non-regularity of \( L_1 = \{ a^i \} \).

Slightly general observation:
for all \( i \geq 0 \), the decomposition \( u, v, w \) gives:
\[ uv^i w = a^{n^3 + (i-1)\cdot n} \cdot a^n \cdot a^{i^3} \cdot a \cdot a \]

However, if such \( y \) existed, then
\[ n^3 + (i-1)\cdot n \leq (n_i)^3 \]
for all \( i \geq 0 \), where \( n_i \in \mathbb{N} \) depends on \( i \).

This is "trivially" false.
Table 5.
(a) \( L = \{ uu^R v \mid u, v \in (01)^+ \} \) is regular, since

\[ L \text{ is denoted by a regular expression} \]

\[ o(ou_1)^+ o \text{ U } 1(ou_1)^+ 1. \]

To see that \( L \subseteq L(0(ou_1)^+ 0 \text{ U } 1(ou_1)^+ 1) \):

Let \( x \in L \) be arbitrary, i.e., \( x = uu^R v \) for some \( u, v \in (01)^+ \).

Since \( u \in (01)^R \), \( u = ou' \) or \( u = 1u' \) for some \( u' \in (01)^* \).

Assume \( u = ou' \) (the case for \( u = 1u' \) is similar).

Then \( x = uu^R v = ou' v (ou')^R = ou' v uu^R o \subseteq o(ou_1)^+ 0 \)

To see that \( L(0(ou_1)^+ 0 \text{ U } 1(ou_1)^+ 1) \subseteq L \):

Let \( x \in L(0(ou_1)^+ 0 \text{ U } 1(ou_1)^+ 1) \) be arbitrary.

Assume \( x \in o(ou_1)^+ 0 \) (the case for \( x \in 1(ou_1)^+ 1 \) is similar).

Then \( x = uu^R v \) where \( u = 0 \) and \( v \in (01)^+ \),

that is, \( x \in L \).

(b) \( L = \{ uu^R v \mid u, v \in (01)\}^+ \) is not regular. Suppose that it were.

We can apply the Pumping Lemma directly on \( L \). Here we use

closure properties for regularity first to "restrict" \( L \) into \( L' \):

Consider \( L' = L \cap \{1(02)^*011 \} \).

Certainly \( L' \) would be regular since "\( \cap \)" preserves regularity.

But, what is \( L' \) (or, why do we consider "\( \cap \)"?)

For \( x \in L' \), \( u^2 v^2 \), \( x \) is of the form \( uu^R v^2 \):

Hence \( L' = \{ 10^{2n+1} 110^{2n+1} 11 \mid n \geq 0 \} \).

Now, apply the pumping lemma

on \( L' \) (remember, there are may cases to check).

\[ \begin{array}{c|c|c}
 2n & 10^{2n+1} & 110^{2n+1} 11 \\
 0 & u & u^R v^2 \\
 1 & \text{an impossible decomposition into } u v^2 v^2 \\
 \end{array} \]

This is the only possible decomposition into \( uu^R v \)

(check this)
Problem 6
(a) We prove $L$ is non-regular by contradiction via the closure properties of regular languages.

Suppose not, $L$ were regular.

Then, $I = \frac{a^* b^3 \ast - L}{(a+b)^*}$ would be regular.

What is $I$?

$I = \{ w \in \{a,b\}^* \mid \#_a(w) = 3 \}$.

Hence, we consider

$I \cap a^* b^* = \{ w \in \{a,b\}^* \mid \#_a(w) = \#_b(w) \land w \to b \text{ no form } a^* b^* \}

= \{ a^i b^j \mid i \geq 3 \}

\text{known non-regular, a contradiction.}

Thus, $L$ is not regular.