1. The examination contains 4 problems. You have 75 minutes for 40 points.

2. Show all important steps in your work. Your answers will be graded on its correctness and clarity.

1. [10 points] Assume the approximation schemes (*) on elementary arithmetic operations on real-valued operands $x$ and $y$:

$$
(x + y)^* = x^* + y^* \quad (x - y)^* = x^* - y^* \\
(x \cdot y)^* = x^* \cdot y^* \quad (x/y)^* = x^*/y^*
$$

Consider $x^* = 0.950$ with its absolute-error bound $|e(x^*)| \leq 0.002$ and $y^* = 0.150$ with its absolute-error bound $|e(y^*)| \leq 0.003$.

(a) [3 points] For $z = x + y$, find an exact upper bound on $|e(z^*)|$. Do the same for $z = x - y$.

(b) [7 points] For $z = x \cdot y$, use the approximation and exact methods studied in this course to find an approximate upper bound and exact upper bound on $|e(z^*)|$. Do the same for $z = x/y$. 


2. [10 points] Consider a (normalized) floating point number system FPNS with base $b = 10$, significand $s = 5$, and the range of exponent: $m = -99, M = +99$. What are the values of:

(a) the smallest positive number in FPNS

(b) the gap length (width of the "chasm") just to the right of zero in FPNS

(c) the second smallest positive number in FPNS

(d) the gap length to the right of the smallest positive number in FPNS

(e) the ratio of the chasm width (part (b)) to the smallest gap length (part (d))

(f) the floating point representation of 1.0 in FPNS

(g) the largest number in FPNS that is less than 1.0

(h) the gap length to the left of 1.0

(i) the smallest number in FPNS greater than 1.0

(j) the gap length to the right of 1.0

(k) the ratio of (the gap length to the right of 1.0) to (the gap length to the left of 1.0)
(l) the smallest positive integer that cannot be represented in FPNS

(m) the smallest representable integer that is greater than the smallest non-representable positive integer, if any such exists

(n) the length of the gap in which the smallest non-representable integer lies, if any such exists

(o) the number of non-representable integers, if any, in the gap in part (n)

(p) the largest number in FPNS

(q) the gap length to the left of the largest number

(r) the number of non-representable integers, if any, in the gap immediately to the left of the largest number
3. [10 points] Consider the function \( f : \mathbb{R} \to \mathbb{R} \) defined by:

\[
f(x) = (1 + x)e^x.
\]

(a) [5 points] Give the degree-3 Taylor polynomial of \( f \) expanded at \( x_0 = 0 \) and its (derivative) remainder term (with proper quantification of any variable appearing in the term). Show all important intermediate work.
(b) [5 points] Give the complete infinite Taylor series of $f$ expanded at $x_0 = 0$ and the coefficient of its $n$-th term.
4. [10 points]

(a) [4 points] Consider the following pseudocode that computes \(n(> 1)\) values of \(x\) equally spaced from \(x = a\) to \(x = b\) (that is, \(x = a, a + h, a + 2h, \ldots, a + (n - 1)h = b\)):

\[
\begin{align*}
  h &:= (b - a)/(n - 1); \\
  x &:= a - h; \quad \text{for } i := 1 \text{ to } n \text{ loop} \\
  &\quad x := x + h; \\
  &\quad \text{-- Use current } x\text{-value ...} \\
  &\quad \text{end loop;}
\end{align*}
\]

Give a precise explanation for the potential pitfalls of the code? Improve the code.
(b) [6 points] Consider the following root-finding formulas for a quadratic equation $ax^2 + bx + c = 0$:

$$
r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}
$$

$$
r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}
$$

When $b^2 \gg ac$ (that is, $b^2$ is sufficiently large than $ac$), give a precise explanation for the potential pitfalls of using the formula(s) above. Improve the root-finding formula(s) through rearranging their expressions.